Assignment 6

This homework is due *Tuesday* Oct 16. Since I forgot to tell about it in class, everyone can take a free one week extension.

There are total 31 points in this assignment. 28 points is considered 100%. If you go over 28 points, you will get over 100% for this homework and it will count towards your course grade (up to 115%).

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper *and give credit to your collaborators in your pledge*. Your solutions should contain full proofs. Bare answers will not earn you much.

This assignment covers section 3.3 in Bartle–Sherbert.

(1) [3pt] (3.3.2) Let $x_1 > 1$ and $x_{n+1} = 2 - 1/x_n$ for $n \in \mathbb{N}$. Show that (x_n) is bounded and monotone. Find the limit.

(2) [3pt] (3.3.3) Lt $x_1 \geq 2$ and $x_{n+1} = 1 + \sqrt{x_n - 1}$ for $n \in \mathbb{N}$. Show that (x_n) is decreasing and bounded below by 2. Find the limit.

(3) [3pt] Find a mistake in the following argument:
"Let (x_n) be a sequence given by $x_1 = 1$, $x_{n+1} = 1 - x_n$. In other words, $(x_n) = (1, 0, 1, 0, 1, 0, \ldots)$. Show that $\lim(x_n) = 0.5$. Indeed, let $\lim(x_n) = x$. Apply limit to both sides of equality $x_{n+1} = 1 - x_n$:

$$\lim(x_{n+1}) = \lim(1 - x_n)$$
$$\lim(x_{n+1}) = 1 - \lim(x_n)$$
$$x = 1 - x,$$

so $x = 0.5$"

(4) [3pt] (3.3.10) Establish convergence or divergence of the sequence (y_n), where

$$y_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \qquad \text{for } n \in \mathbb{N}.$$

(5) (a) [3pt] (Exercise 3.3.11) Let $x_n = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}$, $n \in \mathbb{N}$. Prove that that (x_n) converges. (*Hint*: for $k \geq 2$, $\frac{1}{k^2} \leq \frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k}$.)

(b) [3pt] Let K be a natural number $K \geq 2$. Let $y_n = \frac{1}{1^K} + \frac{1}{2^K} + \frac{1}{3^K} + \cdots + \frac{1}{n^K}$, $n \in \mathbb{N}$. Prove that that (y_n) converges. (*Hint*: compare y_n to x_n.)

(6) (13.3.12) Establish the convergence and find the limits of the following sequences.
(a) [3pt] $((1 + 1/n)^{n+1})$,
(b) [3pt] $((1 + 1/n)^{2n})$,
(c) [3pt] $\left(\left(1 + \frac{1}{n+1}\right)^n\right)$,
(d) [4pt] $((1 - 1/n)^n)$.
(*Hint:* Express these sequences using $X = ((1 + 1/n)^n)$, apply arithmetic properties of limit. In (d), consider the inverse and compare to (c).)